K-homological finiteness of Ruelle algebras

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  - K-homological finiteness of Ruelle algebras

Fredholm modules

Definition (Atiyah, Kasparov)

An odd Fredholm module over a C^*-algebra A is a triple (H, ρ, F) where

- H is a separable Hilbert space;
- $\rho : A \to B(H)$ is a representation;
- $F \in B(H)$ with $F = F^*, F^2 = 1$ and $[F, \rho(a)] \in K(H)$, for all $a \in A$.

An even Fredholm module is the $\mathbb{Z}_2$-graded version.

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The K-homology class $[H, \rho, F] \in \mathrm{K}^*(A)$ gives a pairing map $\mathrm{K}_*(A) \to \mathbb{Z}$;

$$[e] \mapsto \mathrm{Index}(\rho_-(e)F_+\rho_+(e)),$$
$$[u] \mapsto \mathrm{Index}(P\rho(u)P),$$

where $P = (F + 1)/2$ is a projection.

Finite summability

For $p > 0$ the Schatten p-ideal: $L^p(H) = \{T \in K(H) : (s_n(T))_n \in \ell^p(\mathbb{N})\}$.

Definition (Connes)

The Fredholm module (H, ρ, F) over A is p-summable if

$$[F, \rho(a)] \in L^p(H), \quad \text{for a in a dense } * \text{-subalgebra } \mathscr{A}.$$

$\mathscr{A}$ can be assumed stable under holomorphic functional calculus.

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Theorem (Connes Index Formula)

Suppose (H, ρ, F) over A is p-summable on $\mathscr{A}$. Then, by considering K*-theory classes over $\mathscr{A}$ the associated pairing map $\mathrm{K}_*(A) \to \mathbb{Z}$ becomes*

$$[e] \mapsto a_n Tr_s(\rho(e)([F, \rho(e)])^n),$$
$$[u] \mapsto b_n Tr(\rho(u^*)([F, \rho(u)][F, \rho(u^*)])^n [F, \rho(u)]),$$

where $n \in \mathbb{N}$ is even and large enough.

Example

- Represent $A = C(\mathbb{T})$ on $H = L^2(\mathbb{T})$ via ρ as multiplication operators.
- P projects onto the Hardy space $\overline{\text{span}}\{z^n : n \geq 0\}$ and $F = 2P - 1$.
- (H, ρ, F) is a Fredholm module.
  - $[F, \rho(f)]$ is finite rank for trigonometric polynomials f;
  - It is p-summable ($p > 1$) as $[F, \rho(f)] \in L^p(H)$ when $f \in C^\infty(\mathbb{T})$.

Then, for a unitary $u \in C^\infty(\mathbb{T})$ the associated map $\mathrm{K}_1(C(\mathbb{T})) \to \mathbb{Z}$,

$$[u] \mapsto -\frac{1}{2\pi i} \int u^{-1} du,$$

that is, minus the winding number around $0 \in \mathbb{C}$ of u.

Definition (Emerson-Nica, Goffeng-Mesland)

The K-homology of A is uniformly L^p-summable on $\mathscr{A}$ if every $x \in \mathrm{K}^*(A)$ has a Fredholm module representative that is p-summable on $\mathscr{A}$.

Examples

A	$\mathscr{A}$	$p >$	Author(s)
$C(M)$	$C^\infty(M)$	$\dim(M)$	Kasparov, Folklore
CK-algebra	$*$-algebra of generators	0	Goffeng-Mesland
$C(\partial\Gamma) \rtimes \Gamma$	$\mathrm{Lip}(\partial\Gamma, d) \rtimes_{\mathrm{alg}} \Gamma$	$\dim_H(\partial\Gamma, d)$	Emerson-Nica

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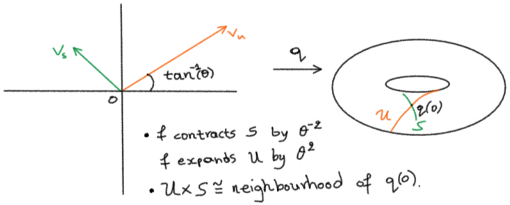
Care is Required

- (Rave) Requiring $\mathscr{A} = A$ leads to degeneracy issues;
- (Puschnigg) For a higher rank lattice Λ, every $0 \neq x \in \mathrm{K}^*(C_r^*(\Lambda))$ is not represented by a finitely summable Fredholm module on $\mathbb{C}\Lambda$;
- (Goffeng-Mesland) There is $0 \neq x \in \mathrm{K}^1(\oplus_{n\in\mathbb{N}} C(S^{2n-1}))$ that does not admit finitely summable Fredholm module representations.

Smale spaces

HTA, slope $\theta = (1+\sqrt{5})/2$

- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with quotient metric via $q : \mathbb{R}^2 \to \mathbb{T}^2$;
- $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ induced by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ acting on $\mathbb{R}^2$;
- $\theta^2 > 1$ is eigenvalue with eigenvector $v_u = (\theta, 1)$.
- $\theta^{-2} < 1$ is eigenvalue with eigenvector $v_s = (-1, \theta)$.



Definition (Ruelle)

A Smale space is a compact metric space (X, d) with a homeomorphism φ so that

- there is $\varepsilon_X > 0$ and a local bi-continuous (bracket) map

$$[\cdot,\cdot] : \{(x,y) \in X \times X : d(x,y) \leq \varepsilon_X\} \to X$$

$$\begin{aligned} [x,x] &= x \\ [x,[y,z]] &= [x,z] \\ [[x,y],z] &= [x,z] \\ \varphi([x,y]) &= [\varphi(x),\varphi(y)]; \end{aligned}$$

- there is $\lambda > 1$ so that for $\varepsilon \leq \varepsilon_X, x \in X$ and

$$\begin{aligned} X^s(x,\varepsilon) &= \{y \in X : d(x,y) < \varepsilon, [x,y] = y\} \\ X^u(x,\varepsilon) &= \{y \in X : d(x,y) < \varepsilon, [x,y] = x\}, \end{aligned}$$

φ and φ^{-1} contract $X^s(x,\varepsilon)$ and $X^u(x,\varepsilon)$ at least by λ^{-1}.

Examples

- Subshifts of finite type. Important subclass:
  - Topological Markov chains
- Smale's nonwandering Axiom A systems. Important subclasses:
  - Anosov diffeomorphisms (e.g Hyperbolic toral automorphisms)
  - Horseshoes from self-affine limit sets
- Certain expansive groups
- Aperiodic substitution tiling spaces
- Fractals from self-similar groups
- Wieler solenoids

Groupoids and C*-algebras (Putnam-Spielberg)

Ruelle, Putnam and Spielberg investigated C^*-algebras from Smale spaces.

For $x \in X$ consider

$$X^s(x) = \{y \in X : d(\varphi^n(x), \varphi^n(y)) \to 0, n \to +\infty\},$$
$$X^u(x) = \{y \in X : d(\varphi^n(x), \varphi^n(y)) \to 0, n \to -\infty\}$$

Let P, Q be periodic orbits. Form equivalence relations on $X^u(Q), X^s(P)$

$$G^s(Q) = \{(x, y) \in X^u(Q) \times X^u(Q) : y \in X^s(x)\},$$
$$G^u(P) = \{(x, y) \in X^s(P) \times X^s(P) : y \in X^u(x)\}.$$

Theorem (Putnam-Spielberg)

The groupoids $G^s(Q)$ and $G^u(P)$ admit an étale structure given by holonomy maps.

To get C^*-algebras, consider the complex vector space $C_c(G^s(Q))$ with convolution and involution

$$(f \cdot g)(v, w) = \sum_{z \in X^s(v)} f(v, z) g(z, w),$$
$$f^*(v, w) = \overline{f(w, v)}.$$

Then, represent $C_c(G^s(Q))$ on the grid-like $H = \ell^2(X^s(P) \cap X^u(Q))$ and get the stable algebra $\mathcal{S}(Q)$ by completion. The unitary $u\delta_y = \delta_{\varphi(y)}$ gives an inner automorphism α_s and hence the stable Ruelle algebra

$$\mathcal{R}^s(Q) = \mathcal{S}(Q) \rtimes_{\alpha_s} \mathbb{Z}.$$

Similarly, represent $C_c(G^u(P))$ on H to get the unstable algebra $\mathcal{U}(P)$ and unstable Ruelle algebra

$$\mathcal{R}^u(P) = \mathcal{U}(P) \rtimes_{\alpha_u} \mathbb{Z}.$$

Examples

- Subshifts of finite type
  - $\mathcal{R}^s(Q)$ is a (stabilised) Cuntz-Krieger algebra.
  - $\mathcal{S}(Q)$ is the (stabilised) AF-core.
- Dyadic solenoid
  - $\mathcal{S}(Q)$ is the (stabilised) Bunce-Deddens algebra of type 2^∞.
- Hyperbolic toral automorphism
  - $\mathcal{S}(Q)$ is a (stabilised) A_θ where θ is an algebraic irrational.

Moreover,

- (Deeley, Goffeng, Spielberg, Strung, Yashinski, Putnam) the stable and unstable algebras are classified up to isomorphism by the Elliott invariant;
- (Putnam-Spielberg, Kirchberg-Phillips) Ruelle algebras are simple, purely infinite and classified up to isomorphism by K-theory.

KK-duality of Ruelle algebras

KK-duality is a noncommutative analogue of Spanier-Whitehead duality. Its definition makes use of KK-theory and the Kasparov product.

Two C^*-algebras A and B are KK-dual if there is a fundamental class in $\mathsf{K}^*(A \otimes B)$ that pairs through Kasparov product with a class in $\mathsf{K}_*(A \otimes B)$, in a way that leads to isomorphisms between the K-theory of A and the K-homology of B, and vice versa.

Theorem (Kaminker-Putnam-Whittaker)

The Ruelle algebras $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ are KK-dual. The fundamental class $\Delta \in \mathsf{K}^1(\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P))$ induces the isomorphism

$$- \otimes_{\mathcal{R}^s(Q)} \Delta : \mathsf{K}_*(\mathcal{R}^s(Q)) \to \mathsf{K}^{*+1}(\mathcal{R}^u(P)).$$

By flipping we also get an isomorphism $\mathsf{K}_*(\mathcal{R}^u(P)) \cong \mathsf{K}^{*+1}(\mathcal{R}^s(Q))$.

The fundamental class $\Delta \in \mathrm{K}^1(\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P))$ is represented by an extension $\tau_\Delta : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to Q(H \otimes \ell^2(\mathbb{Z}))$ built from the representations

- $\rho_s : \mathcal{R}^s(Q) \to B(H \otimes \ell^2(\mathbb{Z}))$

$$a \mapsto \bigoplus_{n \in \mathbb{Z}} \alpha_s^n(a), \quad u \mapsto 1 \otimes B;$$

- $\rho_u : \mathcal{R}^u(p) \to B(H \otimes \ell^2(\mathbb{Z}))$

$$b \mapsto b \otimes 1, \quad u \mapsto u \otimes B^*.$$

which commute modulo $K(H \otimes \ell^2(\mathbb{Z}))$.

Note

The class Δ has a θ-summable Fredholm module representative. Joint work with Mike Whittaker and Joachim Zacharias, to be published soon.

Computation of Kasparov slant product $-\otimes_{\mathcal{R}^s(q)} \Delta$

Let $0 \neq q \in \mathcal{R}^s(Q)$ be a projection and by computing the map $-\otimes_{\mathcal{R}^s(Q)} \Delta$ we get that

- every class in $\mathrm{K}^0(\mathcal{R}^u(P))$ is represented by an even balanced Fredholm module of the form

$$(H \otimes \ell^2(\mathbb{Z}),\, \rho_u,\, \rho_s(u) + 1 - \rho_s(q)), \quad u \in q\mathcal{R}^s(Q)q \text{ is a unitary;}$$

- every class in $\mathrm{K}^1(\mathcal{R}^u(P))$ is represented by an odd Fredholm module of the form

$$(H \otimes \ell^2(\mathbb{Z}),\, \rho_u,\, 2\rho_s(e) - 1), \quad 0 \neq e \in q\mathcal{R}^s(Q)q \text{ is a projection.}$$

Recall that $\rho_s(\mathcal{R}^s(Q))$ and $\rho_u(\mathcal{R}^u(P))$ commute modulo $K(H \otimes \ell^2(\mathbb{Z}))$. Hence, refining this relation yields a refinement of the Fredholm modules.

Groupoid metrics

Theorem (G.)

$G^s(Q)$ and $G^u(P)$ admit a Lipschitz groupoid structure via special metrics D_s and D_u associated to a compatible self-similar metric d' on (X, φ). Also, the groupoid automorphisms from φ are bi-Lipschitz.

Proposition (G.)

The algebraic crossed products

$$\Lambda_s = \mathrm{Lip}_c(G^s(Q), D_s) \rtimes_{\alpha_s, \mathrm{alg}} \mathbb{Z},$$
$$\Lambda_u = \mathrm{Lip}_c(G^u(P), D_u) \rtimes_{\alpha_u, \mathrm{alg}} \mathbb{Z}$$

are well-defined dense $*$-subalgebras of $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$.

Essentially commuting Lipschitz algebras

Proposition (G.)

The algebras $\rho_s(\Lambda_s)$ and $\rho_u(\Lambda_u)$ commute module the Schatten p-ideal $L^p(H \otimes \ell^2(\mathbb{Z}))$, for all $p \gtrsim \dim_H(X, d')$.

This can be strengthened as follows.

Proposition (G.)

There are stable under holomorphic functional calculus $*$-subalgebras $H_s \supset \Lambda_s$ and $H_u \supset \Lambda_u$ so that $\rho_s(H_s)$ and $\rho_u(H_u)$ commute module $L^p(H \otimes \ell^2(\mathbb{Z}))$, for all $p \gtrsim \dim_H(X, d')$.

K-homological finiteness of Ruelle algebras

Theorem (G.)

Let $0 \neq q \in H_s$ be a projection and $p \gtrsim \dim_H(X, d')$. Then,

- *every class in $\mathrm{K}^0(\mathcal{R}^u(P))$ is represented by an even balanced Fredholm module of the form*

$$(H \otimes \ell^2(\mathbb{Z}),\ \rho_u,\ \rho_s(u) + 1 - \rho_s(q)), \quad u \in qH_sq \ \text{is a unitary},$$

  that is p-summable on H_u.
- *every class in $\mathrm{K}^1(\mathcal{R}^u(P))$ is represented by an odd Fredholm module of the form*

$$(H \otimes \ell^2(\mathbb{Z}),\ \rho_u,\ 2\rho_s(e) - 1), \quad 0 \neq e \in qH_sq \ \text{is a projection},$$

  that is p-summable on H_u.

Remarks

- Construct projections and unitaries in the corner subalgebras to get finitely summable Fredholm modules;
- Good knowledge of K-theory would help, e.g the work of Deeley-Yashinski on the K-theory of the stable algebra of a Wieler solenoid;
- Perform index computations;
- Lift these Fredholm modules to spectral triples.

Thank you!

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